

DIRECTION CONTROL OF BILINEAR SYSTEMS. I.

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ABSTRACT. We consider the general bilinear control systems in three-dimensional Euclidean space which satisfy the *LARC* and have an open control set U ; and give sufficient controllability conditions for corresponding projected systems on two-dimensional sphere.

INTRODUCTION AND STATEMENT OF RESULTS

Bilinear control dynamical systems in a Euclidean space $x \in R^n$ are given by linear equations

$$(1) \quad \dot{x}(t) = A(u)x(t), \quad (\Sigma)$$

with the right-hand linear operator $A(u)$ depending linearly on the control parameters $u = (u^1, \dots, u^d) \in U \subset R^d$:

$$(2) \quad A(u) = A + u^1 B_1 + \dots + u^d B_d,$$

where A and $B_k, k = 1, \dots, d$ are some constant linear operators. Then $x(t)$ is the solution (or trajectory) of Σ if

$$(3) \quad \dot{x}(t) = A(u(t))x(t)$$

for some **control** $u(t)$; i.e., the function $u(t) : [0, T] \rightarrow U$. The positive orbit $\mathcal{O}^+(x)$ of the point x is the union of all trajectories $x(t)$ issuing from it; i.e., such that $x(0) = x$. The bilinear system Σ is called **controllable** if for every non-zero point x its orbit coincide with the whole space without its origin zero: $\mathcal{O}^+(x) = R^n \setminus 0$, which means that for two arbitrary non-zero points x, y there always exists some trajectory $x(t)$ of Σ which connect them: $x = x(0), y = x(T)$. We may "project" the bilinear system to the control dynamical system defined on the sphere S^{n-1} of unit vectors in R^n by considering an evolution of unit directions $q(t) = x(t)/\|x(t)\|$ of the non-zero solutions $x(t)$ of Σ : for the unit vector q we define the vector $V(q, u)$ to be equal to the tangent to the sphere component of the vector $A(u)q$; i.e., $V(q, u) = A(u)q - (A(u)q, q)q$. Then

$$(4) \quad \dot{q}(t) = V(q(t), u), \quad (\Sigma^{pr})$$

correctly defines the control dynamical system on the unit sphere which we call the **projection** Σ^{pr} of the system Σ . This definition is justified since for an arbitrary (non-zero) trajectory $x(t)$ with the control $u(t)$ its central projection $x(t) \rightarrow x(t)/\|x(t)\|$ to the unit sphere is the trajectory $q(t)$ with the same control of

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the projected system Σ^{pr} . If in addition some linear operator $A(u_*)$ has all eigenvalues with negative real parts, while some other $A(u^*)$ has all eigenvalues with positive real parts (i.e., the first one takes eventually an arbitrary big sphere inside the unit sphere, while the second - an arbitrary small sphere outside the unit sphere), then the controllability of Σ is equivalent to the controllability of its projection Σ^{pr} : in [CK] (see Corollary 12.2.6) the authors claim that the system Σ is controllable if and only if

- CK1) their Lyapunov spectrum contains zero in its interior, and
- CK2) the projected system Σ^{pr} of Σ is controllable.

Note, that from the linearity of Σ it follows that Σ^{pr} is invariant under the central symmetry $q \rightarrow -q$, and therefore, we may further reduce it to the system on the projective space $RP^{n-1} = S^{n-1}/Z_2$. We do not do this, since for some symmetric and non-controllable systems on the sphere their further reductions to the systems on projective space are controllable; i.e., this further reduction does not preserve simultaneous controllability. It is, also, more convenient for us to work on the unit sphere in the Euclidean space.

Next question is: when the projected system is controllable? The two-dimensional case $n = 2$ was considered in details in [AS] and [BGRS] where the authors show that the projected system on the circle RP^1 satisfying the *LARC* is controllable if and only if for some parameter u_0 the matrix $A + u_0B$ has complex eigenvalues.¹ Remind, that the *LARC* - the "Lie Algebra Rank Condition" is satisfied for the control dynamical system Σ at the point x if the evaluation at this point of the algebra $\mathcal{LA}(\Sigma)$ generated by all vector fields $A(u)x$ coincide with the whole R^n . Easy to see that the systems Σ and Σ^{pr} satisfy or do not satisfy *LARC* simultaneously, and that for $n = 3$ the system Σ^{pr} on the unit two-dimensional sphere $S^2 \subset R^3$ satisfies *LARC* at q if and only if there exists two linearly independent vectors $V(q, u)$ and $V(q, v)$ at this point (our CC1 condition below).

In general, the controllability of the bilinear system (1) corresponds to the transitivity of the Lie algebra generated by the right-hands $A(u)$. Classification of such transitive actions is obtained in [B] (see also [BW], where the algorithm is given for the practical verification of this property). Our approach is quite different: instead of checking the transitivity we study directly the chronological products of the control system (1). The reason is the following. In [M1] we considered quantum systems with Hamiltonian depending on a parameter u . Using the Dirac mechanism of quantization, it was suggested some new description of the Berry-phase geometry for such systems. The problem of measurement of such systems naturally leads to the investigation of dynamical systems with the control u . These systems are defined on "level" sub-manifolds M of the Hamiltonian in the space of parameters; and are not linear systems. We verify the controllability of such systems with the help of the "closed orbit controllability criterium", see [M2], which works for arbitrary non-linear control systems on closed manifolds. For instance, structurally stable dynamical systems on any closed surface M^2 have decompositions into invariant cells with the same dynamics, as we describe below in this paper; and their controllability follows from the same incidence relations. For bilinear systems these relations are extremely simple - they are given by linear inequalities, which is presented in this paper.

More precisely, the main goal of this paper is to give sufficient conditions for controllability of bilinear systems satisfying the *LARC* (CC1 condition) in three-dimensional case and have an open control set U (CC2 condition). We do not consider yet the case $n = 3$ completely in all details, but restrict our attention to the "stable" or "general" systems (see [PDM]) as follows: we choose some finite subset \mathcal{U}^N of parameters $u_k, k = 1, \dots, N$ and assume (our stability or generality conditions CC3, CC4; see below) that the systems of linear equations Σ_k for these parameters have different eigenvalues and their eigenvectors E_i^k are different. Then we define - by a number of linear conditions on these eigenvectors - the open subsets $CO^2(E_i^k)$, which consist of geodesic triangles in the unit sphere. Finally, we prove that the projected system Σ^{pr} is controllable if some simple incidence relations are satisfied, see the Theorems A-C below.

¹For the sake of completeness note that all one-dimensional ($n = 1$) control systems $\dot{x} = A(u)x$ are controllable in our sense iff $A(u) \neq 0$ since all non-zero x have the same direction - the set of unit direction RP^0 is a one-point set.

Our arguments rely on the "Closed Orbit Controllability Criterium" from [M2], where it is proved that the control dynamical system on a compact manifold satisfying the *LARC* and with an open set of control vectors (see definitions in [M2]) is controllable if and only if through an arbitrary point goes some closed trajectory of the system. To make this paper self-content we reprove the two-dimensional version of this criterium here. Then we find sufficient controllability conditions such that compositions of shifts P_k^t along trajectories of some finite number $A(u_k)$ provide us with the collection of closed trajectories going through an arbitrary q .

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1. BILINEAR CONTROL SYSTEMS IN R^3

As we mentioned above, the question if the given bilinear system Σ in R^3 is controllable or not can be reduced to the same question about controllability of the projected system Σ^{pr} on the sphere S^2 . In this section we consider first the more general question of the controllability of systems Σ_M on closed (connected, compact without boundary) surfaces M^2 - we reprove for $n = 2$ the closed orbit controllability criterium from [M2]. Then, using this criterium, we give sufficient conditions for the controllability of the projected systems Σ^{pr} on the two-sphere S^2 . These conditions we formulate in terms of finite control subsystems of Σ^{pr} - for finite number of parameters $u_k, k = 1, \dots, N$ we consider systems

$$(5) \quad \dot{q}(t) = V(q(t), u_k) \quad (\Sigma_k^{pr}),$$

and find conditions on the set of Σ_k^{pr} such that compositions of shifts P_k^t along trajectories of these systems (i.e., chronological products, see [M2]) give us closed trajectories going through all points q of the sphere. This is equivalent to say that every two points q and p may be connected by some trajectory of the system with piece-wise constant control functions with values $u_k, k = 1, \dots, N$.

2. CONTROLLABILITY CONDITIONS

The first condition we require is that the collection of vector fields $V_k(q) = V(q, u_k), k = 1, \dots, N$ satisfies the *LARC* which for S^2 or any other surface M^2 is the following:

CC1. In an arbitrary point $q \in M^2$ the set of all control vectors $\{V(q)\}$ contains two linearly independent vectors.

For the control dynamical system on the surface M^2 the set $V(q) = \{V(q, u) | u \in U\}$ of all control vectors at the point q is the subset of the tangent plane $T_q M^2$ to M^2 at the point q . The union V of all $V(q)$ is the subset of the union of all planes $T_q M^2$ - the tangent bundle TM^2 . The second condition we require is

CC2. The set V is an open subset in the tangent bundle TM^2 .

For projections Σ^{pr} of bilinear systems with an open control set U this second condition CC2 follows from the first one; i.e., from the *LARC*.

Lemma 1. (Closed orbit controllability criterium for $n = 2$) *Let Σ_M be some control dynamical system on the closed connected surface M^2 :*

$$(6) \quad \dot{q}(t) = V(q(t), u).$$

If the system Σ_M satisfies CC1-CC2 above, then it is controllable if and only if through an arbitrary point q goes some closed and non-trivial orbit of the system Σ_M .

Proof. Let B be some coordinate neighborhood of the point q with local coordinate functions $x^i : B \rightarrow R, i = 1, 2$ such that $x^i(q) = 0$. Then the tangent plane to M^2 at the point $q' \in B$ is generated by two coordinate

vectors $e_i(q')$, $i = 1, 2$ - evaluations at this point q' of the coordinate vector fields $e_i = \partial/\partial x_i$, while for the control vectors holds:

$$(7) \quad V(q', u) = V^1(q', u)e_1(q') + V^2(q', u)e_2(q')$$

for some functions $V^i(q', u)$, $i = 1, 2$. If $q(t)$, $0 \leq t \leq T$ is the closed trajectory of Σ_M with the control function $u(t)$ going through the point q , $q = q(0) = q(T)$, then

$$(8) \quad \frac{d}{dt}x^i(q(t)) = V^i(q(t), u(t)).$$

Because CC2 is satisfied, for sufficiently small positive ϵ all vectors V' from $T_{q(t)}M$ for $0 < t < \epsilon$ which are ϵ -close to $V(q(t), u(t))$ belong to the control set $V(q(t))$. Therefore, for an arbitrary small vector $v = (v^1, v^2)$ with $(v^1)^2 + (v^2)^2 < \epsilon^2$ the solution of the equation

$$(9) \quad \frac{d}{dt}x_v^i(t) = V^i(q(t), u(t)) + v^i$$

also gives us the trajectory $q_v(t) = (x_v^1(t), x_v^2(t))$ of the control system Σ_M . Since, for a given $0 < t < \epsilon$ the set of all points $q_v(t)$ is an open set which contains $q(t)$, we see that the set of points $\mathcal{O}^+(q)$ which are reachable from q contains some open neighborhood of (every) $q(t)$ for $0 < t < \epsilon$. By standard theorems on ordinary differential equations, the image of every such open neighborhood under all shifts P^τ , $0 < \tau < T - t$ along trajectories of the system Σ_M with the control function $u(t)$ is an open-neighborhood of the trajectory $q(t)$, $0 < t < T$. Therefore, the positive orbit $\mathcal{O}^+(q)$ of the point q contains together with arbitrary closed trajectory going through q some its open neighborhood - which we denote by $B^+(q)$. The negative orbit $\mathcal{O}^-(q)$ of q is the set of points from which we can reach the point q by trajectories of our system Σ_M . It is by definition the positive orbit of the "minus"- Σ_M control system, i.e., the system $-\Sigma_M$ on M with the control set $-V(q)$. Easy to see that these two systems satisfy or do not satisfy our conditions CC1-CC2 simultaneously. Therefore, we may repeat our arguments and prove that $\mathcal{O}^-(q)$ contains with an arbitrary point $q(t)$ for negative $t < 0$ some its open neighborhood $B^-(q(t))$. Because for our closed orbit it holds $q(t) = q(T + t)$ we see that for arbitrary $0 \leq \tau \leq T + t$ this neighborhood is an image of some open neighborhood of $q(\tau)$ under the shift $P^{T-\tau+t}$ along trajectories of Σ_M . Which implies that there exists some open neighborhood $B^-(q)$ of the closed orbit such that from its every point we can reach q . Then by definition the system Σ_M is controllable on the intersection $B_q = B^-(q) \cap B^+(q)$, which is again an open neighborhood of the closed orbit $q(t)$, $0 \leq t \leq T$.

Now controllability of Σ_M on the connected surface M follows from its compactness, see [M2]. For every $q \in M$ there exists some closed orbit going through this point. Every such orbit has an open neighborhood B_q on which the system Σ_M is controllable. The union of all B_q covers M^2 . Since M^2 is compact there exists some finite sub-covering

$$(10) \quad M^2 = \cup_{i=1, \dots, c} B_{q_i}.$$

Because M^2 is connected for two arbitrary points q and p we may find a finite sequence $B_{q_{i_1}}, \dots, B_{q_{i_k}}$ from the finite covering (10) such that $q \in B_{q_{i_1}}$, $p \in B_{q_{i_k}}$ and $B_{q_{i_s}} \cap B_{q_{i_{s+1}}} \neq \emptyset$. Take $q_0 = q$; some $q_s \in B_{q_{i_s}} \cap B_{q_{i_{s+1}}}$ and $q_{k+1} = p$. Since Σ_M is controllable on each $B_{q_{i_s}}$ there exists a trajectory from q_s to q_{s+1} for all $s = 0, \dots, k$. Taking the composition of these trajectories we obtain the trajectory of Σ_M going from q to p . Since q, p where arbitrary this proves that Σ_M is controllable on M^2 .

To complete the proof of our controllability criterium (... "and only if" part) we note that if the system is controllable than for arbitrary q there exists the trajectory γ of the system going from q to the point $q(-\epsilon)$

for some other non-trivial trajectory $q(t)$, $-\epsilon \leq t \leq \epsilon$ of the system Σ_M going through $q = q(0)$. Then the composition of γ and $q(t)$, $-\epsilon < t \leq 0$ gives us the non-trivial closed trajectory through the point q . The Lemma is proved.

Now we go back to the projected bilinear systems.

For every fixed u_k the system Σ_k^{pr} is the projection of the system of linear differential equations in R^3 :

$$(11) \quad \dot{x}(t) = A(u_k)x(t) \quad (\Sigma_k).$$

Denote by $\lambda_1^k(u)$, $\lambda_2^k(u)$, $\lambda_3^k(u)$ eigenvalues of $A(u_k)$ and by $E_1^k(u)$, $E_2^k(u)$, $E_3^k(u)$ the corresponding unit eigenvectors.

Here we will split our consideration into two cases: first is when all eigenvalues are real, and the second one - when one λ_R^k is real and other two are complex conjugate numbers λ_C^k and $\overline{\lambda_C^k}$.

3. REAL EIGENVALUES. JORDAN CELLS DECOMPOSITION AND DYNAMICS

Here we further assume

CC3. The eigenvalues λ_i^k , $i = 1, 2, 3$ are different:

$$(12) \quad \lambda_1^i < \lambda_2^i < \lambda_3^i,$$

which means that the linear operator $A(u_k)$ is stable - similar to all other operators which are close enough to it. Then in the basis $E_i^k(u)$, $i = 1, 2, 3$ the solutions of the system (11) are given by

$$(13) \quad x_k(t) = (x_k^1(t) = x_k^1(0)e^{\lambda_1^k t}, x_k^2(t) = x_k^2(0)e^{\lambda_2^k t}, x_k^3(t) = x_k^3(0)e^{\lambda_3^k t}).$$

We see that closer of domains in R^3 where coordinates have the same sign are invariant sets of this system. Correspondingly, triangles on the unit sphere with vertices $\pm E_i^k$, $i = 1, 2, 3$ are invariant subsets of the projected system Σ_k^{pr} . Sides of these triangles (also invariant subsets) are intervals of big circles (geodesics) in the unit sphere S^2 . We denote by C_i^k the circle around E_i^k consisting of such intervals; e.g., the circle C_3^k consists of the intervals from E_1^k to E_2^k , then from E_2^k to $-E_1^k$, then from $-E_1^k$ to $-E_2^k$ and from $-E_2^k$ to E_1^k . Inside each invariant triangle the position of the point q may be given with the help of coordinates x_k . For definiteness we consider the triangle $\Delta^k = \Delta^k(+ + +)$ with vertices E_1^k, E_2^k, E_3^k , where all x_k^i are non-negative. For an arbitrary point q inside this triangle its coordinates $(x_k^1(q), x_k^2(q), x_k^3(q))$ in the coordinate system x_k provide homogeneous coordinates $(x_k^1(q) : x_k^2(q) : x_k^3(q))$ for the unit direction q in the projective sphere RP^2 ; it holds $(x_k^1(q))^2 + (x_k^2(q))^2 + (x_k^3(q))^2 = 1$. The vector field $V_k(q) = V(q, u_k)$ of the dynamical system Σ_k^{pr} due to (12) is stable (see [PDM]) and has six zeroes: two sources at $\pm E_1^k$, two saddles at $\pm E_2^k$ and two sinks at $\pm E_3^k$. Circles C_3^k consist of saddle separatrix incoming to saddles $\pm E_2^k$, and circles C_1^k - of outgoing separatrix from saddles to sinks $\pm E_3^k$.

If we denote by P_k^t the shift along trajectories of Σ_k^{pr} - the one-parameter group of diffeomorphisms of S^2 generated by the vector field V_k and given by solutions of (11) - then from (13) (considering mutual ratios of coordinates $x_k^i(t)$ as $t \rightarrow \infty$) we easily conclude the following description of the dynamic of such shifts inside our invariant triangle Δ_k .

Lemma 2.

1.1. For an arbitrary point q in the interior of Δ_k and arbitrary small ϵ -neighborhood $B_1^k(\epsilon)$ of E_1^k there exists a trajectory of Σ_k^{pr} starting at some point $q(\epsilon)$ in $B_1^k(\epsilon)$, going through q and converging to E_3^k .

1.2. For an arbitrary curve $q(t)$, $-\epsilon < t < \epsilon$ which is transversal to the outgoing separatrix $E_2^k E_3^k$ and intersect it at the point $q = q(0)$, and another curve $\bar{q}(\bar{t})$, $-\bar{\epsilon} < \bar{t} < \bar{\epsilon}$ which is transversal and intersect the

incoming separatrix $E_1^k E_2^k$ at the point $\bar{q} = \bar{q}(0)$, such that both $q(t)$ and $\bar{q}(\bar{t})$ belong to the interior of the triangle Δ_k for positive t, \bar{t} , for small enough positive $t < \epsilon'$ there exists a trajectory of Σ_k^{pr} starting at some point $\bar{q}(\bar{t})$ for some positive $\bar{t} = \bar{t}(t)$ and going through the point $q(t)$.

Note that the claims of the Lemma 2 are valid also for non-stable dynamical systems when (12) is wrong, but equal eigenvalues correspond to $A(u_k)$ given by the Jordan form matrix on the subspace generated by corresponding eigenvectors. In other two cases we have or V_k identically zero - when Σ_k is diagonal and has equal eigenvalues; or V_k equals zero on some side of the triangle Δ_k corresponding to the two-dimensional subspace, where the action of $A(u_k)$ is diagonal with equal eigenvalues.

4. COMPLETE SUBSYSTEMS. REAL EIGENVALUES

Let $\mathcal{U}^N = \{u_k | k = 1, \dots, N\}$ be some finite subset of the set U of control parameters, and Σ_k^{pr} as above denotes the projection (11) of Σ_k . We call \mathcal{U}^N **complete** if every two points q, p on the unit sphere can be connected by some trajectory of Σ^{pr} with the piece-wise control function $u(t) : [0, T] \rightarrow \mathcal{U}^N$; i.e., taking values in the finite subset \mathcal{U}^N . Now we give some sufficient conditions which guarantee that \mathcal{U}^N is complete. First we assume in addition to CC3 that our subsystem is general:

CC4. All eigenvectors E_i^k are different, and E_i^k does not belong to any plane $\Pi_{s_1}^j$ generated by $E_{s_2}^j$ and $E_{s_3}^j$ for all different $s_1, s_2, s_3 = 1, 2, 3$.

Next, for an arbitrary point q we generate two sets: first is the set $CO^1(q)$ of saddle points E_2^j , and the second set $CO^2(q)$ - some open subset of the unit sphere, which is the union of (geodesic) triangles in this sphere with sides being big circles. Finally, our completeness criterium is that for some index s both E_1^s and $-E_1^s$ belong to $CO^2(E_3^s)$ and $CO^2(-E_3^s)$, see the Theorem A below. This criterium is given in terms of a number of simple inequalities, which describe mutual positions of systems of eigenvectors $\{E_i^k, i = 1, 2, 3\}$. The construction of the sets CO^1 and CO^2 proceeds as follows.

Construction of CO^1 and CO^2 is by repetition of steps of two types.

First type step.

Take some point q in the interior of Δ_k and assume that some circle C_3^j "cut" this point from the vertex E_3^k of the triangle Δ_k . Then by the Lemma 2 above the trajectory $q(t) = P_k^t(q)$ of Σ_k^{pr} issuing from q converge to E_3^k , and therefore, intersect the circle C_3^j at some point $\bar{q} = P_k^{\bar{t}}(q)$. Then for $t < \bar{t}$ and close to \bar{t} the points $q(t)$ lie in some, say $\Delta_j(++-)$ - the triangle with vertices $E_1^j, E_2^j, -E_3^j$, while for $t > \bar{t}$ and close to \bar{t} the points $q(t)$ lie in the triangle $\Delta_j(+++)$ - the triangle with vertices E_1^j, E_2^j, E_3^j . Applying the Lemma 2 again, now for the triangles $\Delta_j(++-)$ and $\Delta_j(+++)$ we see that with the help of shifts along trajectories of Σ_j^{pr} we may approach arbitrarily close an arbitrary point of the semi-circle consisting of outgoing separatrix from E_2^j - the union of intervals $-E_3^j E_2^j \cup E_2^j E_3^j$, which we denote $Sep_2^j(+)$. Since our trajectories under the flow $P_j^{t'}$ are coming from a neighborhood of the point \bar{q} , they converge to the separatrix only from one side, the points which are reachable from $\bar{q}(t)$ always belong to the same half-sphere relative to C_3^j as the point \bar{q} . Note in addition, that for the given point q in the interior of Δ_k and separated from E_3^k by C_3^j we may find some open neighborhood $B(q)$ such that the same is true for all $q' \in B(q)$. Thus, from the Lemma 2 we conclude the following.

Lemma 3. *If the point q in the interior of Δ_k is separated from E_3^k by C_3^j , then for all q' in some small neighborhood $B(q)$ of q , and an arbitrary point p on $Sep_2^j(+)$ which is not end point of it, i.e., $p \neq \pm E_3^j$ there exist an open neighborhood $B(p)$ such that all points in $B(p)$ are reachable from q' if they belong to the same half-sphere relative to C_3^j as the point \bar{q} .*

Let us write down our conditions in an analytic form. If $S(+)$ and $S(-)$ denote the two half-spheres in which the big circle C divides the unit sphere, then the point q belongs to one of them depending on the sign of the scalar product of q with the vector normal to the plane in which this circle lies. Or, if C lies in the plane Π generated by E_1 and E_2 - the vector normal to Π is the vector product $[E_1, E_2]$, and the points q and p belong to different half-spheres if and only if the corresponding products have different signs:

$$(q, [E_1, E_2])(p, [E_1, E_2]) < 0,$$

or in a more symmetric form:

$$\det(q, E_1, E_2)\det(p, E_1, E_2) < 0,$$

where by $\det(X, Y, Z)$ we denote the mixed product of three vectors X, Y, Z . Define the function: $s(q, C, p) = 1$ if (14) is satisfied, and $s(q, C, p) = -1$ if not. Then the points q and E_3^k are separated by C_3^j if

$$(14) \quad s(q, C_3^j, E_3^k) = 1.$$

The triangle Δ_k is the intersection of three half-spheres out of six $S_i^k(\pm)$, $i = 1, 2, 3$ in which the big circles C_i^k , $i = 1, 2, 3$ - its sides - divide the unit sphere. So, the condition that the point q belong to the interior of Δ_k is that the vector q belongs to the same half-sphere relative to C_i^k as the vertex E_i^k (i.e., which does not belong to this circle):

$$(q, [E_1^k, E_2^k])(E_3^k, [E_1^k, E_2^k]) > 0, (q, [E_2^k, E_3^k])(E_1^k, [E_2^k, E_3^k]) > 0, (q, [E_3^k, E_1^k])(E_2^k, [E_3^k, E_1^k]) > 0,$$

or

$$(15) \quad \det(q, E_1^k, E_2^k)\det(E_3^k, E_1^k, E_2^k) < 0, \det(q, E_2^k, E_3^k)\det(E_1^k, E_2^k, E_3^k) < 0, \det(q, E_3^k, E_1^k)\det(E_2^k, E_3^k, E_1^k) < 0.$$

Again, we define the function $d(q, \Delta) = 1$ if the point q belongs to the interior of the triangle $\Delta = \Delta(E_1, E_2, E_3)$ with vertices E_1, E_2, E_3 ; and $d(q, \Delta) = -1$ - if not. From the above we see that $d(q, \Delta(E_1, E_2, E_3)) = 1$ if and only if

$$(16) \quad s(q, C_i, E_i) = -1 \quad \text{for all} \quad i = 1, 2, 3,$$

where C_i denotes the big circle in the sphere going through the points E_j , $j \neq i$.

If the incidence condition (15) holds for some k , and for this k the separation condition (14) holds for some other j , then we set:

$$(17) \quad E_2^j \in CO^1(q).$$

Second type step.

Let the point $E_2^j \in CO^1(q)$ belongs to some triangle Δ_l , say $\Delta_l(+++)$, such that the semi-circle $Sep_2^j(+)$ cut from this triangle some vertex E_s^l ; i.e., such that end points $\pm E_3^j$ of $Sep_2^j(+)$ lie outside Δ_l . Then, depending on $s = 1, 2, 3$ the shifts $P_l^{t''}$ along trajectories of Σ_l^{pr} of the points p from the chord $\Gamma_l^j = Sep_2^j(+) \cap \Delta_l$ of the semi-circle $Sep_2^j(+)$, which is inside the triangle Δ_l , cover some open subregion Ω_l^j , which we include in $CO^2(q)$ (and define it precisely in a moment below). Since such points p belong to the interior of Δ_l every trajectory $P_l^{t''}(p)$, $0 \leq t'' \leq T''$ can be continued over some open interval $-\epsilon'' < t'' < 0$ such that the extended trajectory contains points from both sides of the chord Γ_l^j . Because all points p from Γ_l^j are interior points of the separatrix $Sep_2^j(+)$ (end points $\pm E_3^j$ of $Sep_2^j(+)$ lie outside Δ_l) we conclude from the Lemma 3 above

that all points of the open domain Ω_l^j are reachable by trajectories from q and from all points q' of some its small neighborhood $B(q)$.

In addition, when Γ_l^j cut the vertex E_s^l for $s \neq 3$ - it has some end point on the incoming separatrix $E_1^l E_2^l$. Therefore, arguing as in the Lemma 3, we also include the limit point of this separatrix E_2^l in CO^1 under this step of the second type.

Write down the analytic form of our arguments. The point E_2^j belong to $\Delta_l(E_1^l E_2^l E_3^l)$ if

$$(18) \quad d(E_2^j, \Delta_l(E_1^l E_2^l E_3^l)) = 1.$$

The endpoints of the semi-circle $Sep_2^j(+)$ do not belong to this triangle if

$$(19) \quad d(\pm E_3^j, \Delta_l(E_1^l E_2^l E_3^l)) = -1.$$

Then the chord Γ_l^j cut from this triangle the vertex E_1^l if

$$(20) \quad s(E_1^l, C_3^j, E_2^l) = 1 \quad \text{and} \quad s(E_1^l, C_3^j, E_3^l) = 1.$$

In this case the union of all trajectories of Σ_l^{pr} issuing from points of Γ_l^j covers the part of Δ_l lying in the same half-sphere relative to C_3^j as E_3^l :

$$(21) \quad \Omega_l^j = \{p | d(p, \Delta_l) = 1, s(p, C_3^j, E_3^l) = -1\}.$$

If the chord Γ_l^j cut from this triangle the vertex E_2^l ; i.e.,

$$(22) \quad s(E_1^l, C_3^j, E_2^l) = 1 \quad \text{and} \quad s(E_2^l, C_3^j, E_3^l) = 1,$$

then the union of all trajectories of Σ_l^{pr} issuing from points of Γ_l^j contain the part of Δ_l lying in the same half-sphere relative to C_3^j as E_2^l - small geodesic triangle with the vertex E_2^l and the opposite side Γ_l^j . Indeed, an arbitrary point p inside this triangle by the Lemma 2 lies on some trajectory $p(t)$ coming from some small neighborhood of E_1^l - outside this triangle, and therefore intersecting Γ_l^j at some point. Hence, the points of

$$(23) \quad \Omega_l^j = \{p | d(p, \Delta_l) = 1, s(p, C_3^j, E_2^l) = -1\}.$$

are reachable from Γ_l^j at this case (22). If the chord Γ_l^j cut from this triangle the vertex E_3^l ; i.e.,

$$(24) \quad s(E_1^l, C_3^j, E_3^l) = 1 \quad \text{and} \quad s(E_2^l, C_3^j, E_3^l) = 1,$$

then similarly all the points from the small triangle with the vertex E_3^l and the opposite to it side Γ_l^j are reachable from the points of this side, and

$$(25) \quad \Omega_l^j = \{p | d(p, \Delta_l) = 1, s(p, C_3^j, E_3^l) = -1\}.$$

After this step we add to $CO^2(q)$ the open region Ω_l^j and to the list $CO^1(q)$ the vertex E_2^l if (20) or (22) holds. The result of the step of the second type we formulate as follows.

Lemma 4. Let $E_2^j \in CO^1(q)$ belongs to some triangle Δ_l , and the semi-circle separatrix $Sep_2^j(+)$ cut from this triangle some vertex E_s^l ; i.e., such that end points $\pm E_3^j$ of $Sep_2^j(+)$ lie outside Δ_l . Then, depending on $s = 1, 2, 3$ the open set Ω_l^j defined by (21) if $s = 1$, (23) if $s = 2$ or (25) if $s = 3$ is contained in the positive orbit $O^+(q')$ for all points q' from some small neighborhood $B(q)$ of q . If $s = 1, 2$ then in addition the conclusion of the Lemma 3 is true for all interior points p of the separatrix $Sep_2^l(+)$.

Now we give our sufficient controllability criterium for the subsystem of $\Sigma_k^{pr}, k = 1, \dots, N$ where all eigenvalues λ_i^k are real.

Theorem A. Let Σ^{pr} be some projected system which satisfies the LARC (CC1), the openness condition CC2) and generality conditions CC3, CC4. Assume that for some finite subset $U^N = \{u_k | k = 1, \dots, N\}$ of U and some index s after some finite number of steps, each of which is of the first or second type as above, it holds

$$(26) \quad E_1^s \text{ and } -E_1^s \text{ belong to } CO^2(E_3^s) \text{ and } CO^2(-E_3^s).$$

Then the system Σ^{pr} is controllable.

Proof. Take an arbitrary point q . As in the lemma 1 we see that, from CC1 and CC2 it follows that both the positive and negative orbits of q contain some open sets - neighborhoods $B^+(q(\epsilon))$ and $B^-(q(-\epsilon))$ of some trajectory $q(t)$ going through q . Thus, taking partition of the unit sphere into the set of triangles $\Delta_s(\pm\pm\pm)$ we see that there is some of these triangles, denote it by Δ_s^+ with the vertex E_3^s or $-E_3^s$ - say E_3^s for definiteness; which intersect the positive orbit of q , say at some point q' . Therefore, by the lemma 2 there exists some trajectory from q' to some p' in an arbitrary small neighborhood of this vertex E_3^s . If we take this small neighborhood to be $B(E_3^s)$ as in the Lemmas 3 and 4 above, then we find a trajectory from q to some point p' from which we can reach all the points of $CO^2(E_3^s)$ and E_1^s and $-E_1^s$ in particular. Since $CO^2(E_3^s)$ is an open set, we can reach all the points from some small neighborhoods of E_1^s and $-E_1^s$, which we denote $B'(E_1^s)$ and $B'(-E_1^s)$.

Since the negative orbit of q also contains some open subset, it has non-empty intersection, say the point q'' , with the interior of some of the triangles Δ_s^- with the vertex E_1^s or $-E_1^s$, say E_1^s for definiteness. Again, by the Lemma 2 the point q'' belongs to some trajectory issuing from some point p'' which we may choose in an arbitrary small neighborhood of this vertex. Finally, the combination of these orbits: from q to q' , then to p' , then to p'' , then to q'' and back to q gives us the closed orbit going through q . Since we choose the point q in an arbitrary way, we have proved that the control dynamical system $\Sigma^k, k = 1, \dots, N$ satisfies the closed orbit controllability criterium and is controllable by the Lemma 1.

The Theorem A is proved.

5. COMPLETE SUBSYSTEMS. COMPLEX EIGENVALUES

When the system Σ_j has one real eigenvalue $\lambda_1^j = \lambda_R^j$ and the pair of complex-conjugate eigenvalues $\lambda_2^j = Re(\lambda_C^j) + \sqrt{-1}Im(\lambda_C^j)$ and $\lambda_3^j = Re(\lambda_C^j) - \sqrt{-1}Im(\lambda_C^j)$ then its solutions are given by $x_j(t) = (x_j^1(t), x_j^2(t), x_j^3(t))$ where

$$x_j^1(t) = x_j^1(0)e^{\lambda_R^j t}$$

and

$$x_j^2(t) = x_j^2(0)\cos(Im(\lambda_C^j)t) + x_j^3(0)\sin(Im(\lambda_C^j)t)e^{Re(\lambda_C^j)t},$$

$$x_j^3(t) = -x_j^2(0)\sin(Im(\lambda_C^j)t) + x_j^3(0)\cos(Im(\lambda_C^j)t)e^{Re(\lambda_C^j)t}.$$

Here we have only two two-dimensional invariant subsets - half-spaces $R_+^3 = \{x_j^1 \geq 0\}$ and $R_-^3 = \{x_j^1 \leq 0\}$. Correspondingly, the projected system Σ_j^{pr} has two invariant cells which are half-spheres. They intersect along the circle C^j which is also an invariant set of Σ_j^{pr} . Another invariant set - two zeros $\pm E_1^j$ of V_j . If

$$(27) \quad \lambda_R^j > \operatorname{Re}(\lambda_C^j),$$

then zeroes $\pm E_1^j$ are sources of the dynamical system Σ_j^{pr} , while C^j is an attracting cycle, and if

$$(28) \quad \lambda_R^j < \operatorname{Re}(\lambda_C^j),$$

then zeroes $\pm E_1^j$ are sinks of the dynamical system Σ_j^{pr} , while C^j is a repulsive cycle. The analog of the Lemma 2 describing the dynamic of Σ_k^{pr} is the following.

Lemma 5.

1. If E_1^j is the source, and C^j is an attracting cycle, then for an arbitrary point q in the interior of Δ_j and arbitrary small ϵ -neighborhood $B(E_1^j, \epsilon)$ of E_1^j there exists a trajectory $q(t)$ of Σ_j^{pr} starting at some point $q(\epsilon)$ in $B(E_1^j, \epsilon)$, going through q and converging to the big circle C^j : for an arbitrary point p of this circle and arbitrary q there exists a sequence $t_n(q, p) \rightarrow \infty$ such that $q(t_n)$ converge to the point p .

2. Let E_1^j be the sink, C^j - the repulsive cycle, and $p(\bar{t})$ some curve intersecting transversally C^j at the point $p = p(0)$. Then for an arbitrary point q in the interior of Δ_j and arbitrary small ϵ -neighborhood $B(p, \epsilon)$ of p there exists a trajectory $q_{\bar{t}}(t)$ of Σ_j^{pr} starting at the point $p(\bar{t})$ in $B(p, \epsilon)$, going through q and converging to the sink E_1^j .

Using arguments from the Theorem A above we see that the LARC (CC1) and the openness (CC2) conditions with the help of the Lemma 5 immediately imply that the positive orbit of the source E_1^j (under (27)) contains some open neighborhood of the half-sphere $\Delta^j(+)$ (and $\mathcal{O}^+(-E_1^j) \supset \Delta^j(-)$ correspondingly); while if E_1^j is a sink (when (28) holds) even more is true: the positive orbit of any small curve $p(\bar{t})$ intersecting C^j transversally at this point p contains the whole unit sphere.

Having at least one such Σ_j^{pr} in our finite subsystem simplifies our controllability considerations: Let \mathcal{U}_R^N denotes the parameters u_k in \mathcal{U}^N such that Σ_k has only real eigenvalues, and $CO^1(q)$, $CO^2(q)$ constructed as above for the subsystem \mathcal{U}_R^N . Then repetition of the arguments in the proof of the Theorem A give us the following.

Theorem B. Let Σ^{pr} be some projected system which satisfies the LARC (CC1), the openness condition CC2) and generality conditions CC3, CC4; and $\mathcal{U}^N = \{u_k | k = 1, \dots, N\}$ some finite set of parameters such that for one of them j the system Σ_j has complex eigenvalues. Assume that for the subset $\mathcal{U}_R^N = \{u_k | k = 1, \dots, N\}$ of parameters such that all eigenvalues of Σ_k are real, and some index s after some finite number of steps, each of which is of the first or second type as above, it holds:

$$(29) \quad E_1^j \text{ and } -E_1^j \text{ belong to } CO^2(E_3^s) \text{ and } CO^2(-E_3^s),$$

if (27) is satisfied; or

$$(30) \quad S^j \text{ intersects both } CO^2(E_3^s) \text{ and } CO^2(-E_3^s),$$

when (28) holds. Then the system Σ^{pr} is controllable.

The most simple case is when we have two systems Σ_j and Σ_l with complex eigenvalues with different types of dynamics: when Σ_j^{pr} has the attracting cycle S^j , while Σ_l^{pr} has the repulsive cycle S^l . Then, using the system Σ_j^{pr} with attracting cycle we may construct a trajectory from an arbitrary point q to an arbitrary point of the attractive cycle S^j - say, to its intersection with the repulsive cycle S^l , and then continue this trajectory further to an arbitrary point with the help of the system Σ_l^{pr} . We have the following controllability condition.

Theorem C. *Let Σ^{pr} be some projected system which satisfies the LARC (CC1), the openness condition CC2) and generality conditions CC3, CC4; and in the finite set $\mathcal{U}^N = \{u_k | k = 1, \dots, N\}$ of parameters for one of them u_j the system Σ_j has complex eigenvalues satisfying (27), and for another u_l the system Σ_l has complex eigenvalues satisfying (28). Then the system Σ^{pr} is controllable.*

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